THE WALKING PARALLEL PAIR HAS SIFTED COLIMITS

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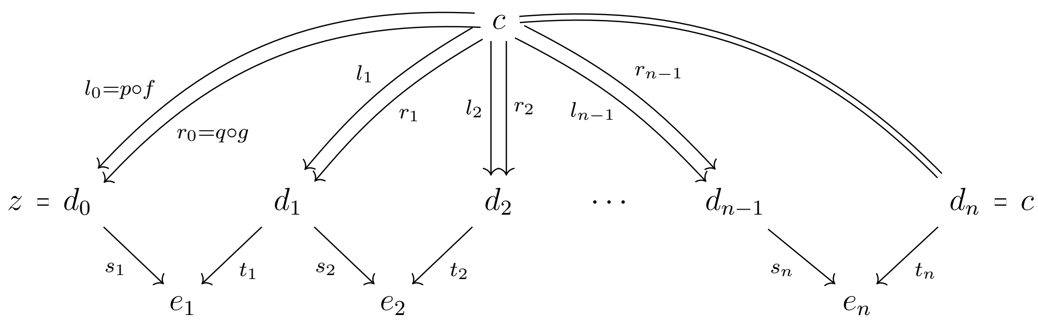
Let $D\colon \mathcal{C} \to \{u, v\colon 0 \rightrightarrows 1\}$ be a functor from a sifted category. If the object 1 is not contained in the image under D, the object 0 gives a colimit of D because the sifted category $\mathcal{C}$ is connected. In what follows, we suppose that there is an object $c_0 \in \mathcal{C}$ such that $D(c_0) = 1$.

We first claim that for every object $c \in \mathcal{C}$ such that $D(c) = 0$, there is a morphism $f\colon c \to x$ with $D(x) = 1$; moreover, which of u and v such a morphism is sent to by D is independent of the choice of f. The existence of f is easy. Indeed, since $\mathcal{C}$ is sifted, there is a cospan $c \to x \leftarrow c_0$, and $D(x) = 1$ follows from $D(c_0) = 1$.

To show the independence of the value of $D(f)$, suppose that there are morphisms $f\colon c \to x$ and $g\colon c \to y$ such that $D(f) = u$ and $D(g) = v$. Since $\mathcal{C}$ is sifted, there is a cospan consisting of $p\colon x \to z$ and $q\colon z \leftarrow y$. Since $\mathcal{C}$ is sifted again, two cospans $(p \circ f, q \circ g)$ and $(\mathrm{id}_c, \mathrm{id}_c)$ are connected to each other, that is, there are a zigzag consisting of $s_i\colon d_{i-1} \to e_i$ and $t_i\colon e_i \leftarrow d_i$ $(1 \le i \le n)$ and parallel pairs $(l_i, r_i)\colon c \rightrightarrows d_i$ $(0 \le i \le n)$ such that $d_0 = z$, $l_0 = p \circ f$, $r_0 = q \circ g$, $d_n = c$, $l_n = r_n = \mathrm{id}_c$, $s_i \circ l_{i-1} = t_i \circ l_i$, and $s_i \circ r_{i-1} = t_i \circ r_i$ $(1 \le i \le n)$.



Then, the equality $D(t_1) \circ D(l_1) = D(s_1) \circ D(l_0) = u$ implies that either $D(l_1) = u$ or $D(t_1) = u$ holds, while $D(t_1) \circ D(r_1) = D(s_1) \circ D(r_0) = v$ implies that either $D(r_1) = v$ or $D(t_1) = v$ holds. However, the only possible combination is $D(l_1) = u$ together with $D(r_1) = v$, and by repeating this argument, we have $D(l_n) = u$ and $D(r_n) = v$, which is a contradiction.

By the claim, each object $c \in \mathcal{C}$ can be classified exclusively into the following three cases:

(1) $D(c) = 1$;
(2) $D(c) = 0$ and there is a morphism from itself sent to u by D;
(3) $D(c) = 0$ and there is a morphism from itself sent to v by D.

Now, we have a cocone $(\alpha_c\colon D(c) \to 1)_{c \in \mathcal{C}}$ under D by letting $\alpha_c \coloneqq \mathrm{id}_1$ if c is classified into the first case, $\alpha_c \coloneqq u$ for the second case, and $\alpha_c \coloneqq v$ for the third case. Moreover, this is a unique cocone under D: If β is another cocone, its vertex should be 1 by the existence of c_0. If $c \in \mathcal{C}$ is classified into the first case, β_c should be the identity. For the second case, taking a morphism $f\colon c \to x$ such that $D(f) = u$, we can obtain $\beta_c = \beta_x \circ D(f) = D(f) = u$. Similarly, we have $\beta_c = v$ for the third case. This concludes $\beta = \alpha$, and since there is no non-trivial endomorphism on the vertex 1, α gives a colimit.

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